Physics 195 / Applied Physics 195 — Assignment #2

Professor: Donhee Ham
Teaching Fellows: Brendan Deveney and Laura Adams
Date: Sep. 20, 2017
Due: 12:45pm + 10 min grace period, Sep. 29, 2017 at the dropbox outside Maxwell-Dworkin Room 131.

Several parts of Problem #4 would require the complete knowledge of Lecture #5, which we will be finishing on Sep. 27. So I will just move Problem #4 below to the next Assignment. So for this Assignment, you don't have to do Problem #4. The due remains the same.

Problem 1 (100 pt): Kinetic inductance of a 3D conductor

(a) A 3D conductor with a cross sectional area A and a length l (along the x-axis) contains a total of N conduction electrons. $n_0 = N/(Al)$ is the conduction electron density per unit volume. m is the electron mass. Show that the kinetic inductance L_K of the conductor along the length is given by

$$L_K = \frac{l}{A} \cdot \frac{m}{n_0 e^2}, \tag{1}$$

by resorting to the Drude equation of motion.

(b) Let's re-derive L_K of Eq. (1) from the Fermi gas viewpoint. Set $T = 0$ K. Ignore electron scattering.[1] A constant electric field, applied along the length starting at $t = 0$, moves all electrons along the x direction with the acceleration identical for all electrons. In the $\vec{k}$-space, this acceleration corresponds to a temporal k_x-value shift that is identical for all electrons. Let Δk_x be that shift at $t = \Delta t$ with respect to $t = 0$; $i.e.$, the Fermi sphere is shifted by Δk_x along the k_x-direction by $t = \Delta t$, as compared to $t = 0$.

- Argue that the total kinetic energy of electrons acquired by time $t = \Delta t$ due to the acceleration by the electric field is given by

$$U = g \times \frac{Al}{(2\pi)^3} \int_{\Sigma'} \epsilon(\vec{k}) d^3\vec{k} - g \times \frac{Al}{(2\pi)^3} \int_{\Sigma_0} \epsilon(\vec{k}) d^3\vec{k}, \tag{2}$$

  where Σ_0 and Σ' respectively signify the Fermi sphere at $t = 0$ and $t = \Delta t$, $g = 2$ accounts for the spin degrees of freedom, and $\epsilon(\vec{k}) = (\hbar^2 k^2)/(2m)$ is the single electron energy. Carry out this calculation to the leading non-vanishing order of Δk_x.

- Argue that the current created due to the acceleration by the electric field is given, at time $t = \Delta t$, by

$$I = g \times \frac{A}{(2\pi)^3} \int_{\Sigma'} e v_x(\vec{k}) d^3\vec{k} \tag{3}$$

  where $v_x(\vec{k}) = \hbar k_x/m$. Carry out this integration (no approximation is needed).

- From Eqs. (2) and (3), you will see $U \sim I^2$. From this, find L_K and check its agreement with Eq. (1).

(c) Repeat Part (b) now for a general T, $i.e.$, $without$ assuming $T = 0$. Electrons are distributed in the $\vec{k}$-space according to the Fermi-Dirac statistics $f(\epsilon(\vec{k}))$, and this distribution is shifted by the applied electric field. All the rest thoughts flow just as in Part (b). To re-derive L_K of Eq. (1), you will first have to argue that U and I at $t = \Delta t$ (again with the corresponding k_x-value shift of Δk_x for all electrons) are now given by

$$U = g \times \frac{Al}{(2\pi)^3} \int \epsilon(\vec{k}) f(\epsilon(\vec{k} - \Delta k_x \hat{n})) d^3\vec{k} - g \times \frac{Al}{(2\pi)^3} \int \epsilon(\vec{k}) f(\epsilon(\vec{k})) d^3\vec{k} \tag{4}$$

$$I = g \times \frac{A}{(2\pi)^3} \int e v_x(\vec{k}) f(\epsilon(\vec{k} - \Delta k_x \hat{n})) d^3\vec{k}, \tag{5}$$

---

[1] The kinetic inductance, which is due to electron acceleration, has the same value whether or not electron scattering is considered.

where $\hat{n}$ signifies a unit vector along the k_x-axis and each of the three integrals is taken over the entire $\vec{k}$ space.

Problem 2 (50 pt): Kinetic inductance of a 2D conductor

Repeat Part (c) of Problem 1 for a 2D conductor to derive the 2D kinetic inductance

$$L_K = \frac{l}{W} \times \frac{m}{n_{0,2D}e^2} \tag{6}$$

where $n_{0,2D}$ is the conduction electron density per unit area, W is the cross sectional width of the 2D conductor, and l is the length of the 2D conductor.
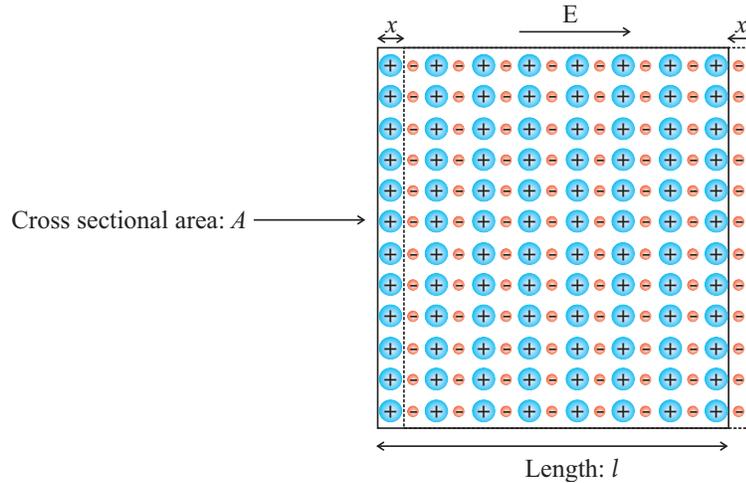
Problem 3 (50 pt): Bulk plasma oscillation

Figure 1: Bulk plasma oscillation

(a) Consider a bulk metal with a cross-sectional area of A and a length of l [Fig. 1]. If all conduction electrons are shifted to right (along the length direction) by x as shown in Fig. 1, surface charges will develop on both sides of the metal—the left side with positive surface charges (positive background ions fixed in the crystal lattice), and the right side with negative surface charges (conduction electrons). These surface charges will produce an electric field across the metal and will pull and accelerate electrons back to their equilibrium distribution. But electrons will not stop at their original positions due to their inertia that has acquired kinetic energy via the acceleration, and they will overshoot to the left. Then the restoring force is reversed and the process repeats itself. This leads to collective oscillation of electrons, or *plasma oscillation*. By calculating the restoring electric force on electrons as a function of x (you can use Gauss's law) and writing down Newton's equation of motion, calculate the plasma oscillation frequency ω_p (angular frequency) in terms of electric permittivity $\mathcal{E}_0$, conduction electron density n_0, electron mass m, and electron charge e. Ignore electron scattering (which will give rise to the damping of the plasma oscillation) in this calculation.

(b) The plasma oscillation can be alternatively described by treating the metal as an $L_K C$ circuit, where the kinetic inductance L_k represents the inertial acceleration of electrons and the parallel plate capacitor C is associated with the Coulomb restoring force described above. Evaluate C (we already know L_K from Problem 1). Show that the plasma oscillation frequency ω_p of Part (a) can be alternatively obtained via

$$\omega_p = \frac{1}{\sqrt{L_K C}}. \tag{7}$$

(c) Now we take into account electron scatterings by resorting to the Drude model with a mean electron scattering time of τ. How is the characteristic damping time of the plasma oscillation related to τ? Express

the quality factor Q of the plasma oscillation in terms of ω_p and τ. If $Q \ll 1$, plasma oscillation is masked by damping; if $Q \sim 1$, plasma oscillation starts becoming observable; and of course if $Q \gg 1$, plasma oscillation dominates over the damping or Ohmic behavior. Calculate $f_p \equiv \omega_p/(2\pi)$ and Q for gold, silver, and copper, and comment on whether their bulk plasma oscillations are observable at room temperature.

Several parts of Problem #4 right below would require the complete knowledge of Lecture #5, which we will be finishing on Sep. 27. So I will just move Problem #4 below to the next Assignment. So for this Assignment, you don't have to do Problem #4.

Problem 4 (100 pt): 1D metal with tight binding

Consider the 1D crystal of Lecture Note #5—a crystal length L, a unit cell length a, 1 atom per unit cell, and a total number of atoms N with $L = Na$—in the tight-binding approximation. As we studied, in the tight-binding approximation, the matrix element of the Hamiltonian H for an electron in the periodic potential due to the 1D crystal lattice may be written as

$$\langle n|H|m \rangle = \epsilon_0 \delta_{nm} - t[\delta_{n-1,m} + \delta_{n+1,m}] \tag{8}$$

in the basis of atomic ground states $\{|n\rangle\}$ ($n = 1, 2, 3, \cdots, N$). ϵ_0 and t (> 0) are as we defined in class/note. Finally, assume that there is one valence electron per atom.

(a) Show that a single-electron state with a state index k

$$|\psi_k\rangle = A \sum_{n=1}^{N} e^{-ikna}|n\rangle \tag{9}$$

is a single-electron energy eigenstate (A: normalization constant) and calculate the corresponding single-electron energy eigenvalue $\epsilon(k)$. Determine the normalization constant A. Check the orthogonality of the energy eigenstates, $i.e.$, $\langle \psi_k|\psi_{k'}\rangle = 0$ for $k \neq k'$ (you may find Poisson summation formula useful).

(b) Calculate the density of states $D(\epsilon)$.

(c) Argue that this crystal is a metal. How much total electronic energy lowering is achieved by brining together the N atoms into the 1D crystal? That is, what is the cohesive energy contributed by the valence electrons? For this calculation, assume $T = 0$.

(d) Calculate the chemical potential $\mu(T)$ and the total electronic energy $U(T)$ both to the second order of T, assuming $k_B T \ll t$. From the expression of $U(T)$, calculate the specific heat $C(T)$.

(e) Calculate the effective electron mass m^* near $ka \ll 1$.